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Solitons in a nonlinear model medium

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Abstract. The periodic stationary solutions of a model nonlinear evolution equation simulating the propagation of short-wave perturbations in a relaxing medium are studied. Solutions expressed by a multiple-valued function are shown to exist. A method for determining the nonlinear interaction between solitary waves is suggested. An example of a collision of solitons is given.

The paper deals with periodic solutions and soliton solution to the evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0. \tag{1}$$

It is possible to obtain this model equation describing the short-wave perturbations in a relaxing medium [1] when the equations of motion are closed by the dynamic equation of state [2, 3]. The variable u is the dimensionless pressure. In a relaxing medium, neglecting nonlinear effects, weak short waves obey the linear Klein–Gordon equation [4]. Taking account of nonlinearity caused by the wave propagation rate dependence on the amplitude leads, after certain transformations—factorization and shift in space with small perturbations velocity—to the equation (1).

The equation under study is nonlinear and contains a purely dispersive term (this is shown by the dispersion relation of a linearized equation (1) $\omega=-k^{-1}$). There is a certain analogy with the Korteweg–de-Vries (KdV) equation. Equation (1) and the KdV equation have the same hydrodynamic nonlinearity, but different dispersion terms. This gives hope that (1) may at least partially possess the remarkable properties inherent to the KdV equation, soliton solutions included.

Equation (1) is related to that of Whitham [5, section 13.4] with a kernel $K(x)=\frac{1}{2}|x|$ [1], namely

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{2}\int_{-\infty}^{\infty}|x-s|\frac{\partial u}{\partial s}\,\mathrm{d}s=0. \tag{2}$$

It should be noted the Whitham equation possesses interesting properties; in particular it describes solitary wave-type formations, has periodic solutions and explains the existence of the limiting amplitude [5]. An important property is the presence of conservation laws for waves decreasing rapidly at infinity.

We succeed in integrating (1) and finding the stationary solutions for travelling waves. The solution is looked for in the form

$$u(x,t)=u(x-vt)=u(\eta) \qquad \eta=x-vt. \tag{3}$$

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We thus pass from two independent variables to a single one, parameter v. Let us carry out calculations for waves decreasing rapidly at infinity. After substitution of (3) into (2) and one-fold integration we have

$$-vu+\tfrac{1}{2}u^2+\hat{K}u+c=0. \tag{4}$$

By definition

$$\hat{K}u=\tfrac{1}{2}\int_{-\infty}^{\infty}|\eta-s|u(s)\,\mathrm{d}s.$$

Making use of the fact that

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}|\eta|=2\delta(\eta)$$

so that the following relation is valid

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\hat{K}u=u$$

and employing the operator $\mathrm{d}^2/\mathrm{d}\eta^2$ in (4) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(u-v)^2+2u=0. \tag{5}$$

This equation is obtained for waves decreasing rapidly at infinity. However the same equation is valid also for periodic solutions. The integration in this case should be performed over the period of the solution [6]. It should be noted that equation (5) may be obtained from (1) by substituting (3). Then by writing $z=u-v$ we reduce the above equation to the form

$$\frac{\mathrm{d}}{\mathrm{d}\eta}zz'+(z+v)=0.$$

The latter, being multiplied by zz', is integrated as

$$\tfrac{1}{2}(zz')^2=-\tfrac{1}{3}z^3-\tfrac{1}{2}vz^2+c_1.$$

The prime denotes, as usual, the derivative. The trinomial in the RHS is conveniently expressed in terms of its zeros a_1, a_2 and a_3. Thus

$$\tfrac{1}{2}(zz')^2=-\tfrac{1}{3}(z-a_1(z-a_2)(z-a_3). \tag{6}$$

It is easy to verify that if there are complex roots the value z tends to minus infinity, and this contradicts the physical statement of the problem. Indeed, if we have only one real root, the graph of the function

$$f(z)=-\tfrac{1}{3}(z-a_1)(z-a_2)(z-a_3)$$

composed of the RHS of (6), crosses the z axis once. Thus as $z\to+\infty$ we have $f\to-\infty$ and as $z\to-\infty$ we have $f\to+\infty$. But since the trinomial in the integration region should always be positive, as follows from the LHS of (6), this region extends in z from minus infinity to the value of the single real root. This means the perturbation amplitude $u=z+v$ also tends to minus infinity, which does not correspond to the physical statement of the problem. So, all roots of the trinomial should be real. This requires that c_1 should be between $\frac{1}{6}v^3$ and 0. For definiteness we shall assume that $a_1\leqslant a_2\leqslant a_3$.

Since there is only one constant c_1 a single root only can be chosen at one's discretion, say a_3. The other roots are related via v by the formulae

$$a_{2,1}=\tfrac{1}{2}(-q\pm\sqrt{q^2-4a_3q}) \qquad q\equiv\tfrac{3}{2}v+a_3.$$

Then always the root $a_3\in[0,0.5v]$ for $v>0$ or $a_3\in[-v,-1.5v]$ for $v<0$. It follows from the above formula that always $a_1<0$, and the root a_2 changes its sign depending on that of v.

There are two cases to consider, the first when $v>0$ and $a_2<0$ and the second when $v<0$ and $a_2>0$. The integration region of (6) is the interval (a_2, a_3) on which $f>0$. At the points $z=a_2$ and $z=a_3$ the derivatives are zero. We integrate equation (6) to obtain

$$\pm\sqrt{\tfrac{2}{3}}\,\eta+c_2=\int_z^{a_3}\frac{z\,\mathrm{d}z}{\sqrt{(z-a_1)(z-a_2)(a_3-z)}}$$
$$=2a_1F(\varphi,k)/\sqrt{a_3-a_1}+2\sqrt{a_3-a_1}\,E(\varphi,k). \qquad (7a)$$

Here $F(\varphi,k)$, $E(\varphi,k)$ are incomplete elliptic integrals of the first and second kind, respectively, $k=\sqrt{(a_3-a_2)/(a_3-a_1)}$ and

$$\varphi=\sin^{-1}\sqrt{(a_3-z)/(a_3-a_2)}. \qquad (7b)$$

The constant c_2 is determined by the initial phase of a wave profile; without loss of generality it may be set zero. The relations (7) give a parametric representation of z as a function of η in the form $z=z(\varphi)$, $\eta=\eta(\varphi)$.

For a wave moving with velocity $v>0$ there exists a singularity at $z=0$ (this point is in the integration region $a_2\leqslant z\leqslant a_3$) where the derivatives z' go to infinity. In the vicinity of $z=0$ the solution obeys the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(z^2+v\eta^2)=0$$

i.e. the integral curve passes over the ellipse centred on the line $u=v$. This testifies to the ambiguity of the functional dependence $u=u(\eta)$. Graphs of the amplitude u versus the coordinate η are given in figure 1. The solutions are periodic. Besides that, in

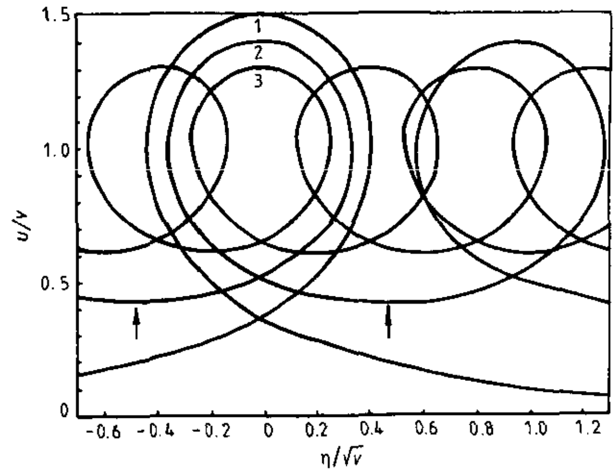

Figure 1. Stationary waves at $v>0$.

certain regions they are ambiguous. The curves are shaped as periodically repeated loops. For the limiting amplitude $u_{max} = 1.5v$ (then $c_1 = \frac{1}{6}v^3$) the periodic wave degenerates into a solitary wave (curve 1 in figure 1).

When the wave moves with velocity $v < 0$, the point $z = 0$ is not in the integration interval (a_2, a_3). The solution $u = u(\eta)$ (figure 2) is always unambiguous. At small amplitudes the wave is transformed into a sinusoidal one with a period $2\pi\sqrt{|v|}$. As the maximum amplitude increases, the period decreases insignificantly. The wave profile is smooth if the limiting maximum amplitude is not attained. For a wave with limiting amplitude $u_{max} = 0.5|v|$ the profile character undergoes changes. Curve 1 corresponding to this case consists of parabolas. At $\eta = 3\sqrt{|v|}$ (this is a half-period) the function is sharp and the derivative $du/d\eta = -\sqrt{|v|}$ changes its sign.

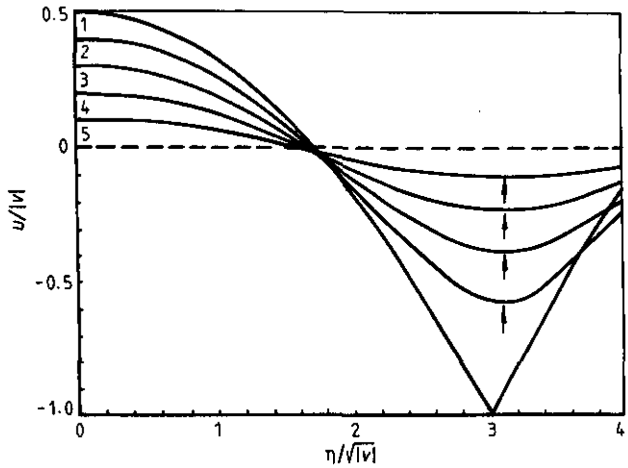

Figure 2. Periodic waves at $v < 0$.

Solitary waves exist only when $v > 0$. In figure 1 this is shown by curve 1. The formulae (7) expressing the parametric dependence of u on η are simplified in this case. Taking account of (3) the solution for solitary waves is written as

$$u = \tfrac{3}{2}v \operatorname{sech}^2 \frac{\chi}{2\sqrt{v}}$$
$$x - vt = \chi - 3\sqrt{v} \tanh \frac{\chi}{2\sqrt{v}}. \tag{8}$$

The value χ plays the role of the parameter in these dependences.

The ambiguous solution can be given a physical interpretation. The wave perturbation destroys the thermodynamical equilibrium (the dynamical processes) while the interaction between particles of the medium aspires to restore this equilibrium (relaxation processes). In the case considered the relaxation time is slow compared to the characteristic time of the wave field change, so that particles fail to interact one with another. Consequently particles with different thermodynamic characteristics will be in one microvolume.

A study of the interaction between solitary waves with different v proves to be of interest. At the same time the ambiguity of the functional dependence $u = u(x)$ for one solitary wave imposes difficulties in a direct numerical integration of equation (1). The unambiguity of both functions (8) u and x of the parameter χ proves to be helpful in advancing the solution of this problem.

We consider the interaction between two solitary waves moving in the same direction with velocities v_1 and v_2, so that $v_1 > v_2$. A solitary wave decreases exponentially at infinity. Let us dispose two waves in such a way that, at the initial time moment, the centre of one of them was at the point $x_1 = 0$ and the centre of the other was at $x_2 > 0$, their interaction being negligibly small ($x_2/\sqrt{v_i} \gg 1$). Before times when the waves produce mutual influence, the solution can be represented as a superposition of solutions and can be written in terms of one parameter μ:

$$u = u_1 + u_2$$

$$u_1 = \tfrac{3}{2}v_1 \operatorname{sech}^2 \frac{\mu - v_1 t}{2\sqrt{v_1}} \qquad (9)$$

$$u_2 = \tfrac{3}{2}v_2 \operatorname{sech}^2 \frac{\mu - \mu_2 - v_2 t}{2\sqrt{v_2}}$$

$$x = \mu - 3\sqrt{v_1} \tanh \frac{\mu - v_1 t}{2\sqrt{v_1}} - 3\sqrt{v_2}\left(1 + \tanh \frac{\mu - \mu_2 - v_2 t}{2\sqrt{v_2}}\right). \qquad (10)$$

It is clear that there is a relation $x_2 = \mu_2 - 3(\sqrt{v_1} + \sqrt{v_2})$ and at $\mu = 0$ we have $x = x_1 = 0$. Here it was taken into account that $\mu_2/\sqrt{v_i} \gg 1$.

It should be noted that we manage to choose the coordinates in which the initial equation is linear and thus the wave interaction is studied easily since it is a superposition of two solutions. Such coordinates are

$$\mathrm{d}\xi = \mathrm{d}x - u\,\mathrm{d}t \qquad \tau = t \qquad (11)$$

where equation (1) has the form

$$\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \tau} u\right) + u = 0. \qquad (12)$$

Obviously, the solution for a single solitary wave in these coordinates takes the following parametric form:

$$u_1 = \tfrac{3}{2}v_1 \operatorname{sech}^2 \frac{\mu - v_1 \tau}{2\sqrt{v_1}} \qquad (13)$$

$$\mathrm{d}\xi = (1 - u_1/v_1)\,\mathrm{d}\mu. \qquad (14)$$

Note that the functional dependence of u_1 on time τ (13) has become unambiguous. The transformation (11) is analogous to the transition from Euler coordinates to Lagrange ones. Thus, the time is unambiguous for a particle. Certainly, this fact plays a positive role in solving the problem posed. It should also be noted that in (14) the dependence on $\mathrm{d}\tau$ vanishes. As (12) is linear the interaction between the waves u_1 and u_2 is expressed as a superposition

$$u = u_1 + u_2 = \tfrac{3}{2}v_1 \operatorname{sech}^2 \frac{\mu - v_1\tau}{2\sqrt{v_1}} + \tfrac{3}{2}v_2 \operatorname{sech}^2 \frac{\mu - \mu_2 - v_2\tau}{2\sqrt{v_2}}$$

through the parameter μ. It remained to determine ξ and then x through μ. It is easy to verify by direct substitution that if we put

$$\mathrm{d}\xi = \left(1 - \frac{u_1^2/v_1 + u_2^2/v_2}{u_1 + u_2}\right)\mathrm{d}\mu \tag{15}$$

the linearity condition is satisfied, namely

$$\frac{\partial}{\partial\xi}\left\{\frac{\partial}{\partial\tau}(u_1+u_2)\right\} + (u_1+u_2) = 0.$$

In the relation (15) time is a parameter. The integration of (15) at fixed time taking into account (11) gives the dependence of x on μ. In the coordinates x, t we have the solution

$$\begin{aligned} u &= \tfrac{3}{2}v_1 \operatorname{sech}^2 \frac{\mu - v_1 t}{2\sqrt{v_1}} + \tfrac{3}{2}v_2 \operatorname{sech}^2 \frac{\mu - \mu_2 - v_2 t}{\sqrt{v_2}} \\ x &= x_0 + \int \left(1 - \frac{u_1^2/v_1 + u_2^2/v_2}{u_1 + u_2}\right)\mathrm{d}\mu. \end{aligned} \tag{16}$$

The constant x_0 for an arbitrary time t is determined by the condition $x = \mu + 3\sqrt{v_1}$ at $x \to -\infty$. Thus, we obtained the solution as parametric dependences (16) on the parameter μ, the functions obtained being unambiguous.

The parametric solution (16) was computed for the case $v_2 = 0.5\ v_1$. The solution is given in figure 3. For convenience of representation the observer moves relative to the initial system with constant velocity $v = 0.5(v_1 + v_2)$. The waves then move one onto another. The waves converge and become deformed. The solitary wave with smaller amplitude loses its form faster than the larger-amplitude solitary wave. The smaller wave is absorbed by the larger one. At the moment when the solitary waves are coincident the resulting wave is symmetric in the cordinate η. When these solitary

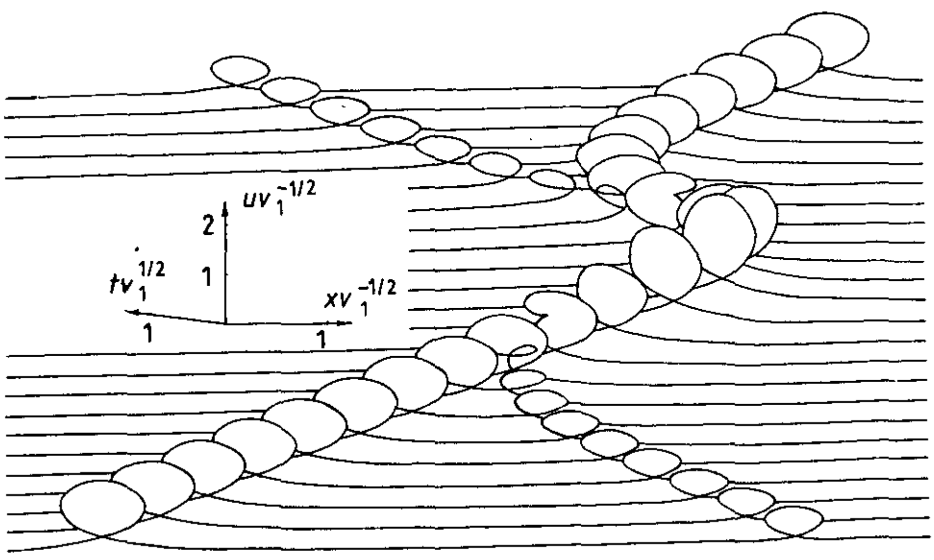

Figure 3. Interaction between solitons v_1 and $v_2 = 0.5v_1$.

waves are sufficiently far apart they have the form and velocity of the initial solitary waves. After the interaction the solitary wave with velocity v_1 is displaced backwards a distance $6\sqrt{v_2}$ while the solitary wave with velocity v_2 is displaced forwards a distance $6\sqrt{v_1}$. These phaseshifts can also be obtained from equation (10). Indeed, at $t=0$ the centres of the waves were at points $x_1=0$ and $x_2=\mu_2-3(\sqrt{v_1}+\sqrt{v_2})$ and at $t \gg \mu/(v_1-v_2)$, as follows from (10), $x_1=v_1 t-6\sqrt{v_2}$ and $x_2=\mu_2+v_2 t+3(\sqrt{v_1}-\sqrt{v_2})$, respectively. Consequently after the interaction the waves are shifted as mentioned above.

At each of the solitary waves preserves its shape and velocity after interaction, each exhibits the distinguishing property of a soliton. We conclude that the solitary wave given by (8) is a soliton.

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